Maechler 2012

$$\mathrm{log1pexp}(x) = \begin{cases} \exp(x) & x \leqslant x_0 = -37 \\ \mathrm{log1p}(\exp(x)) & -37 < x \leqslant x_1 = 18 \\ x + \exp(-x) & x_1 < x \leqslant x_2 = 33.3 \\ x & x > x_2 \end{cases}$$

These bounds are computed for Float64 only. Let's compute generic bounds.

For x_0, compute relative error between $\log(1+\mathrm{e}^x)$ and e^x:

$$\Delta_0 = \frac{\mathrm{e}^x - \log(1+\mathrm{e}^x)}{\log(1+\mathrm{e}^x)} < \mathrm{e}^x$$

for $x < 0$. Therefore $\Delta_0 < \varepsilon$ whenever $x < \log(\varepsilon) = x_0$.

For x_1, compute relative error between $\log(1+\mathrm{e}^x)$ and $x + \mathrm{e}^{-x}$:

$$\Delta_1 = \frac{x + \mathrm{e}^{-x} - \log(1+\mathrm{e}^x)}{\log(1+\mathrm{e}^x)} = \frac{\mathrm{e}^{-x} - \log(1+\mathrm{e}^{-x})}{\log(1+\mathrm{e}^x)} < \frac{\frac{\mathrm{e}^{-2x}}{2}}{\log(1+\mathrm{e}^x)} < \frac{\mathrm{e}^{-2x}}{2}$$

provided $\log(1+\mathrm{e}^x) > 1$, that is $x > \ln(\mathrm{e}-1)$, and where we used the alternating series $\log(1+\mathrm{e}^{-x}) > \mathrm{e}^{-x} - \frac{\mathrm{e}^{-2x}}{2}$. Therefore $\Delta_1 < \varepsilon$ whenever $x > -\frac{1}{2}\ln(2\varepsilon) = x_1$.

For x_2, compute relative error between $\log(1+\mathrm{e}^x)$ and x:

$$\Delta_2 = \frac{\log(1+\mathrm{e}^x) - x}{\log(1+\mathrm{e}^x)} = \frac{\log(1+\mathrm{e}^{-x})}{\log(1+\mathrm{e}^x)} < \log(1+\mathrm{e}^{-x})$$

provided $x > \ln(\mathrm{e}-1)$. Then $\Delta_2 < \varepsilon$ whenever $x > -\ln(\mathrm{e}^\varepsilon - 1)$.

A tighter x_2 can be found by:

$$\Delta_2 = \frac{\log(1+\mathrm{e}^{-x})}{\log(1+\mathrm{e}^x)} < \frac{\mathrm{e}^{-x}}{x}$$

Solving $\frac{\mathrm{e}^{-x}}{x} = \varepsilon$ gives $x_2 = \mathcal{W}(1/\varepsilon)$, where $\mathcal{W}(\cdot)$ is Lambert's function. For large argument Lambert's function has the asymptotic form:

$$x_2 = -\log(\varepsilon) - \log(-\log(\varepsilon)) - \frac{\log(-\log(\varepsilon))}{\log(\varepsilon)} + \cdots$$

An even tighter x_2 can be found by solving numerically $\frac{\log(1+\mathrm{e}^{-x})}{\log(1+\mathrm{e}^x)} = \varepsilon$.