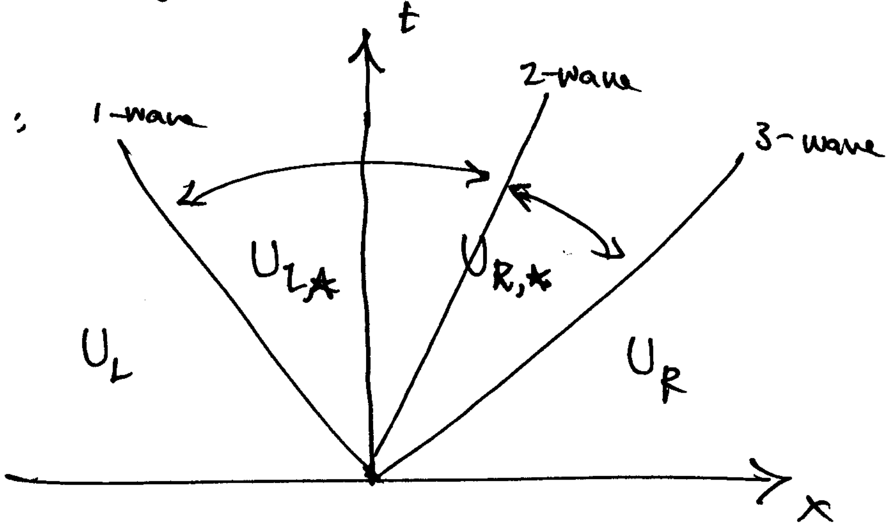
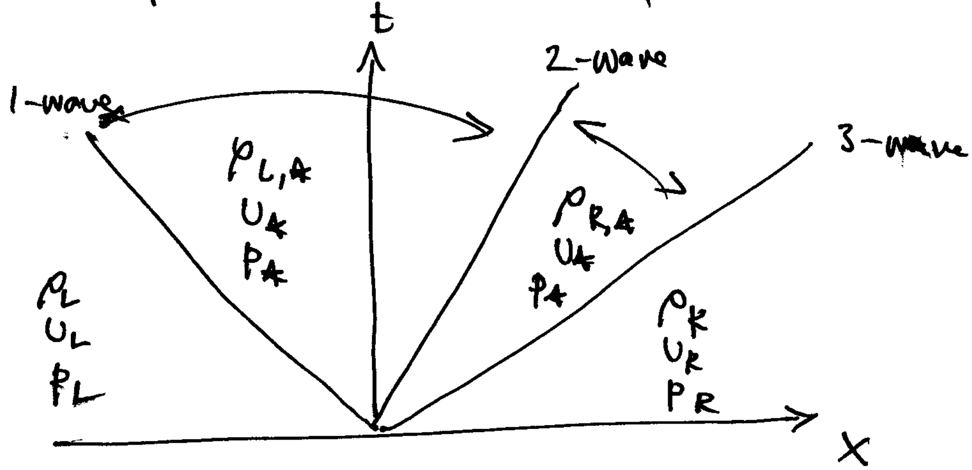
Exact Riemann

(following Colella & Glaz and our PPM+EOS paper)

We have:



In terms of primitive variables, it is



the first goal is to find the star state, by solving

$$u_{L,\star}(p_\star) - u_{R,\star}(p_\star) = 0 \qquad (1)$$

velocity connecting L-state and $\star$-state across the 1-wave

velocity connecting R-state and $\star$-state across the 3-wave

Waves 1 & 3 can be either a shock or rarefaction

Van Leer showed that Newton's method applied to (1) gives:

$$u_{*,s}^{\nu} = u_s \pm \frac{p_*^{\nu} - p_s}{W_s^{\nu}}$$

(CG eq 19) (2)

$$p_*^{\nu+1} = p_*^{\nu} - \frac{Z_L Z_R (u_{*,R} - u_{*,L})}{Z_L + Z_R}$$

where ν is the iteration index and

$$Z_s = \left| \frac{dp_*}{du_{*s}} \right|$$ ← slope in the p–u plane

and W_s is the Lagrangian wave speed

$$W_s = \begin{cases} \dfrac{|p_* - p_s|}{|u_{*,s} - u_s|} & \text{if } u_{*,s} \neq u_s \\ \underbrace{C_s}_{\text{Lagrangian sound speed}} & \text{if } u_{*,s} = u_s \end{cases}$$

(CG eq. 16)

* Note for a shock, we need to find W_s by solving the R-H jump conditions

The form of Z depends on shock or rarefaction.

For rarefaction,

$$Z_s = \left| \frac{dp_*}{du_{*,s}} \right| = C(p_*, \rho_*)$$

← essentially the Riemann invariant

This can be integrated.

Integrating Z (for rarefaction)

$$\frac{d\tau}{dp} = -C^{-2}$$

$$\frac{du}{dp} = \pm C^{-1}$$

+ for 1-wave

− for 3-wave

(these are essentially

$l \cdot q$ — characteristic

eigenvector vars)

from $U = U_s$ to $p = p_*$

Integration is straightforward and then

$$Z = C(p_*, \rho_*)$$

is simply computed from the end state.

We can also take W_s from CG Eq 16

Solving the shock constraint for W^2

$$W^2(e_* - e_s) = \frac{1}{2}(p_*^2 - p_s^2)$$

$$W^2\left[e\left(p_*, \tau_s - \frac{p_* - p_s}{W^2}\right) - e_s\right] = \frac{1}{2}(p_*^2 - p_s^2)$$

define $f(W) \equiv W^2\left[e\left(p_*, \tau_s - \frac{p_* - p_s}{W^2}\right) - e_s\right] - \frac{1}{2}(p_*^2 - p_s^2)$

then do Newton's method:

note: units of W are $\frac{g}{cm^3}\frac{cm}{s} = \frac{g}{cm^2 s}$

$$f(W_0 + \delta W) \approx f(W_0) + f'(W_0)\delta W$$

$$\delta W = -\frac{f(W_0)}{f'(W_0)}$$

then

$$\frac{\partial f}{\partial W} = 2W\left[e(p_*, \tau_*) - e_s\right] + W^2 \left.\frac{\partial e}{\partial \rho_*}\right|_p \frac{\partial \rho}{\partial W}$$

now

again, jump condition

$$\frac{\partial \rho}{\partial W} = \frac{\partial}{\partial W}\left[\frac{1}{\rho_s} - \frac{[p]}{W^2}\right]^{-1}$$

$$= -\left[\frac{1}{\rho_s} - \frac{[p]}{W^2}\right]^{-2}\left(2W^{-3}[p]\right) = -2W^{-3}[p]\rho_*^2$$

For a shock, we have (from jump conditions)

$$Z_s = \left| \frac{dp_*}{du_{*,s}} \right| = \frac{W_s^2}{W_s - (dW/dp_*)[p]}$$

we know W_s

CG Eq 23 gives $\frac{dW^2}{dp_*}$

CG drops 's' subscript

$$\frac{dW^2}{dp_*} = \frac{(C_*^2 - W_s)}{\bar{p}\,p_0 - p_\tau} \frac{W^2}{[p]}$$

$\left(\text{note: } \frac{dW^2}{dp_*} = 2W \frac{dW}{dp_*}\right)$

In this form, we are assuming and EOS in the form $p = p(\tau, e)$

Across the shock, we have

$$W^2 [e] = \frac{1}{2} [p^2]$$

follows from CG eq 12, combining and noting $[p^2] = 2\bar{p}\,[p]$

We also have from the EOS

$$e_* = e\Big(p_*, \underbrace{\tau - \frac{[p]}{W^2}}_{\tau_*}\Big)$$

These can be combined into a single non-linear equation for $\underline{\underline{W^2}}$

We can't simply do Eq 16, here instead we are solving the jump conditions

6.

We need several thermodynamic derivatives

for the dW^2/dp expression, we need

$$p_e = \left.\frac{\partial p}{\partial e}\right|_\rho$$

$$p_\tau = \left.\frac{\partial p}{\partial \tau}\right|_e$$

for the NR procedure to find W, we need

$$\left.\frac{\partial e}{\partial \rho}\right|_p$$

We want to express these in terms of $p(\rho, T)$, $e(\rho, T)$

$$\left.\frac{\partial p}{\partial e}\right|_\rho = \left.\frac{\partial p}{\partial T}\right|_\rho \left.\frac{\partial T}{\partial e}\right|_\rho = \left.\frac{\partial p}{\partial T}\right|_\rho \left(\left.\frac{\partial e}{\partial T}\right|_\rho\right)^{-1}$$

(by considering $p(T(\rho, e), \rho)$)

$$\left.\frac{\partial p}{\partial \rho}\right|_e = \left.\frac{\partial p}{\partial \rho}\right|_T + \left.\frac{\partial p}{\partial T}\right|_\rho \left.\frac{\partial T}{\partial \rho}\right|_e$$

if we take $de = \left.\frac{\partial e}{\partial \rho}\right|_T d\rho + \left.\frac{\partial e}{\partial T}\right|_\rho dT = 0$, then

$$\left.\frac{\partial T}{\partial \rho}\right|_e = - \left.\frac{\partial e}{\partial \rho}\right|_T \Big/ \left.\frac{\partial e}{\partial T}\right|_\rho$$

We also derived these in the Castro paper

7.

$$\left.\frac{\partial e}{\partial \varphi}\right|_p \quad \text{can be obtained via} \quad e(\rho, T(\varphi, p))$$

$$\left.\frac{\partial e}{\partial \varphi}\right|_p = \left.\frac{\partial e}{\partial \rho}\right|_T + \left.\frac{\partial e}{\partial T}\right|_\rho \left.\frac{\partial T}{\partial \varphi}\right|_p$$

then $dp = \left.\frac{\partial p}{\partial \varphi}\right|_T d\rho + \left.\frac{\partial p}{\partial T}\right|_\varphi dT = 0$

$$\therefore \left.\frac{\partial T}{\partial \varphi}\right|_p = - \left.\frac{\partial p}{\partial \varphi}\right|_T \Big/ \left.\frac{\partial p}{\partial T}\right|_\varphi$$

so

$$\left.\frac{\partial e}{\partial \varphi}\right|_p = \left.\frac{\partial e}{\partial \varphi}\right|_T - \left.\frac{\partial e}{\partial T}\right|_\varphi \left.\frac{\partial p}{\partial \varphi}\right|_T \left(\left.\frac{\partial p}{\partial T}\right|_\rho \right)^{-1}$$

This gives us everything we need to compute the star state.

For each wave (1 and 3), we determine if p_* indicates we are a shock or rarefaction and then compute the corresponding W_s, Z_s

Then we can solve system (2) to get an updated p_*

We keep iterating until we converge.

To kick things off, we need an initial guess for p_*.
We can use the 2-shock expression from Toro

$$p_* = \left[(W_r p_l + W_l p_r) + W_l W_r (u_l - u_r) \right] \frac{1}{W_l + W_r}$$

and simply take $W_l = C_l = \sqrt{\Gamma_{1,l} p_l \rho_l}$ and likewise for W_r

Finally, we sample the solution

We need the wave speeds.

For the contact, we have $\lambda = u_\star$

For a rarefaction, there is a head and a tail

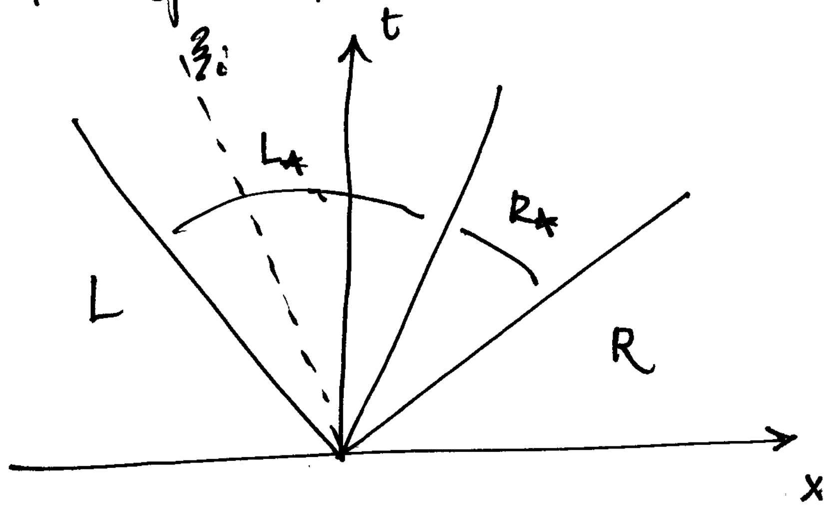
left:
$$\lambda_{head} = u_L - c_s$$
$$\lambda_{tail} = u_{\star,\star} - c_{s,\star}$$

right:
$$\lambda_{head} = u_R + c_s$$
$$\lambda_{tail} = u_\star + c_{s,R}$$

For a shock, the speed comes from the jump conditions

$$\lambda = \underset{L,R}{u_s} \mp \frac{W_s}{\rho_s}$$

The only complication comes in the case when the rarefaction spans the axis

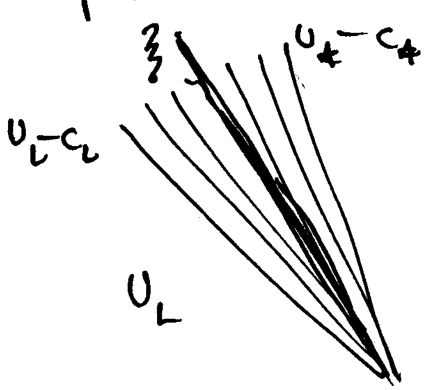


imagine we want the solution along ξ_i

$$\xi_i = \frac{x_i - x_c}{t} \qquad \text{where } x_c = \frac{1}{2}(x_{max} - x_{min})$$

this is define so $\xi_i = 0$ is the center.

A complication arises if $\lambda_{head} < \xi_i < \lambda_{tail}$



not U_L or U_*, but something inbetween

in this case, we integrate from $U = U_L$ to $\xi = u - c$

Following CG, we define

$$\chi = \mathrm{sign}(\xi - u_\star)$$

$\chi < 0 \rightarrow$ we care about $L, L_\star$ states

$\chi > 0 \rightarrow$ we care about $R, R_\star$ states

Previously, for the Riemann invariants, we integrated

$$\frac{du}{dp} = -\frac{1}{C} \qquad \frac{d\tau}{dp} = -\frac{1}{C^2}$$

but now, we need to stop based on u, so we change to

$$\frac{dp}{du} = -C \qquad \frac{d\tau}{du} = \frac{1}{C}$$

then our stopping condition is

$$u_{stop} = \begin{cases} \xi + c(\tau, p) & \text{left rarefaction} \\ \xi - c(\tau, p) & \text{right rarefaction} \end{cases}$$

but we can't evalute this at the start, since $c = c(\tau, p)$, so we need to recompute u_{stop} each integration step.