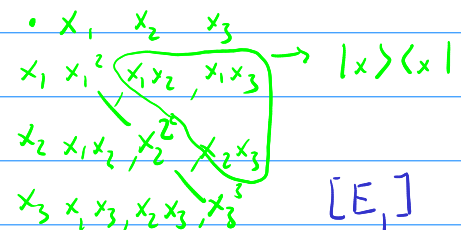


Balanced Maxcut

The one constraint is that the number of nodes assigned to Group 1 must equal to half the number of nodes. In graphs with an odd number of nodes, their group size differences must differ by at most one.

Let $x_i \in \{0, 1\}$ denote group membership of the i^{th} node. Let N denote the number of nodes. Group and node indexing are zero-based; there is a Group 0 and $i \in 0 \dots (N-1)$. The penalty term for this constraint is thus

$$\begin{aligned} & \left(\sum x_i - \frac{N}{2} \right)^2 \\ &= \left(\sum x_i \right)^2 - 2 \left(\sum x_i \right) \left(\frac{N}{2} \right) + \left(\frac{N}{2} \right)^2 \\ &= \left(\sum x_i \right)^2 - N \sum x_i + \frac{N^2}{4} \end{aligned}$$



The first term evaluates to a double sum. Diagonal entries are converted to linear terms via the identity $x_i^2 = x_i$ in QUBO.

$$\left(\sum x_i \right)^2 = \sum_i \sum_j x_i x_j = \underbrace{\sum_i x_i}_{\text{diagonal}} + 2 \underbrace{\sum_{i < j} x_i x_j}_{\text{triangular symm.}} \quad [E_2]$$

It follows that the constraint in QUBO terms are:

$$\left(\sum_i x_i + 2 \sum_{i < j} x_i x_j \right) - N \sum x_i + \frac{N^2}{4} \quad [E_1 | E_2]$$

$$\begin{aligned} & 2 \sum_{i < j} x_i x_j + \underbrace{(1-N) \sum x_i}_{N \text{ linear}} + \underbrace{\frac{N^2}{4}}_{\text{constant}} \quad [E_2] \\ & 2A + (1-N)B + \frac{N^2}{4} \end{aligned}$$

We will use the Ising model to perform the quantum procedure with IonQ and Qiskit. We need to deal with terms A and B separately.

$$A = \sum \sum x_i x_j = \sum \sum \overset{\text{Ising}}{\frac{1}{2}(I - z_i)} \overset{\text{Ising}}{\frac{1}{2}(I - z_j)}$$

$$= \frac{1}{4} \sum \sum (I - z_i)(I - z_j)$$

$$= \frac{1}{4} \sum \sum \underbrace{(I^2 - I z_j - z_i I + z_i z_j)}_{\text{FOIL}}$$

$$= \frac{1}{4} \sum \sum I - \frac{1}{4} \sum \sum (z_i + z_j) + \frac{1}{4} \sum \sum z_i z_j$$

collect expand.

$$= \frac{1}{4} \frac{N(N-1)}{2} I - \frac{N+1}{4} \sum z_i + \frac{1}{4} \sum \sum z_i z_j$$

$$2A = \frac{1}{2} \frac{N(N-1)}{2} I - \frac{N+1}{2} \sum z_i + \frac{1}{2} \sum \sum z_i z_j$$

prev. triangular z_i appears $2(N-1)$ times
 $\frac{1}{2} I$ $-2 \frac{N-1}{2} = -(N-1) = +(-N)$

$$B = \sum x_i = \sum \frac{1}{2}(I - z_i) = \frac{1}{2} \sum (I - z_i)$$

$$= \frac{1}{2} \sum I - \frac{1}{2} \sum z_i$$

$$= \frac{N}{2} I - \frac{1}{2} \sum z_i$$

$$(1-N)B = \frac{N(1-N)}{2} I - \frac{1-N}{2} \sum z_i$$

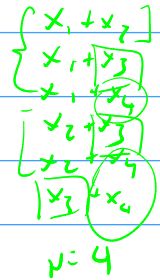
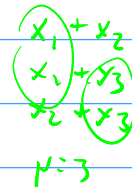
$-I$

In terms of the Z basis, our Ising model is thus

$$\frac{1}{2} \frac{N(N-1)}{2} I - \frac{N+1}{2} \sum z_i + \frac{1}{2} \sum \sum z_i z_j - \frac{N(1-N)}{2} I - \frac{1-N}{2} \sum z_i + \frac{N^2}{4}$$

$$\left(\frac{1}{2} - 1\right) \frac{N(N-1)}{2} I - \left(\frac{N+1}{2} + \frac{1-N}{2}\right) \sum z_i + \frac{1}{2} \sum \sum z_i z_j + \frac{N^2}{4}$$

$$\frac{1}{2} \frac{N(N-1)}{2} \rightarrow -\frac{N^2 - N}{4} = -\frac{N^2}{4} + \frac{N}{4} \quad \frac{N}{2}$$



There will be $\frac{N(N-1)}{2}$ edges. At least $N-2$ edges will be used for the graph. They are used to test connectivity. Only connections within groups can be used - no between-group connections nor disconnections can be used.

The ansatz must be changed to accommodate more qubits. There will be $N + \frac{N(N-1)}{2} = N + \frac{N^2}{2} - \frac{N}{2} = \frac{N(N+1)}{2}$ qubits. Each edge is indexed by two numbers with the second strictly greater than the first. The entries of the H matrix is 1 iff $(u,v) \notin E$. Also, for every pair, it contributes to the parity if $x_u \neq x_v$. This means no disconnections nor intergroup edges can be used.

$$\sum \sum [(u,v) \notin E] e_{ij}$$

$$\sum \sum [x_u \neq x_v] e_{ij} = \sum [1 - x_u - x_v + 2x_{ij}] e_{ij}$$

$$= \sum e_{ij} - \sum x_i e_{ij} - \sum x_j e_{ij}$$

$$\forall u \sum_v e_{uv} \geq 1$$

$$+ \underbrace{\sum x_i x_j e_{ij}}_{\text{cubic}}$$