Balanced Maxcut

The one constraint is that the number of nodes assigned to Group 1 must equal to half the number of nodes. In graphs with an odd number of nodes, their group size differences must differ by at most one.

Let $x_i \in \{0\ 1\}$ denote group membership of the i^{th} node. Let N denote the number of nodes. Group and node indexing are zero-based; there is a Group 0 and $i \in 0..(N-1)$. The penalty term for this constraint is thus

$$
\begin{aligned}
&\left(\sum x_i - \frac{N}{2}\right)^2 \\
&= \left(\sum x_i\right)^2 - 2\left(\sum x_i\right)\left(\frac{N}{2}\right) + \left(\frac{N}{2}\right)^2 \\
&= \left(\sum x_i\right)^2 - N\sum x_i + \frac{N^2}{4}
\end{aligned}
\qquad [E_1]
$$

$$
\begin{array}{c|ccc}
\cdot & x_1 & x_2 & x_3 \\
x_1 & x_1^2 & x_1x_2, & x_1x_3 \\
x_2 & x_1x_2, & x_2^2, & x_2x_3 \\
x_3 & x_1x_3, & x_2x_3, & x_3^2
\end{array}
\rightarrow |x\rangle\langle x|
$$

The first term evaluates to a double sum. Diagonal entries are converted to linear terms via the identity $x_i^2 = x_i$ in QUBO.

$$
\left(\sum x_i\right)^2 = \sum_i \sum_j x_i x_j = \underbrace{\sum_i x_i}_{\text{diagonal}} + \underbrace{2\sum_i \sum_{j=i+1} x_i x_j}_{\text{triangular symm.}} \qquad [E_2]
$$

It follows that the constraint in QUBO terms are:

$$
\left(\sum_i x_i + 2\sum_i \sum_{j=i+1} x_i x_j\right) - N\sum x_i + \frac{N^2}{4} \qquad [E_1 | E_2]
$$

$$
\underbrace{2\sum_i \sum_{j=i+1} x_i x_j}_{\frac{1}{2}N(N-1)\ \text{triangular}} + \underbrace{(1-N)\sum x_i}_{N\ \text{linear}} + \underbrace{\frac{N^2}{4}}_{\text{constant}} \qquad [E_2]
$$

$$
2A + (1-N)B + \frac{N^2}{4}
$$

We will use the Ising model to perform the quantum procedure with IonQ and Qiskit. We need to deal with terms A and B separately.

$$A = \sum\sum x_i x_j = \sum\sum \underbrace{\tfrac{1}{2}(I-Z_i)}_{\text{Ising}}\underbrace{\tfrac{1}{2}(I-Z_j)}_{\text{Ising}}$$

$$= \underbrace{\tfrac{1}{4}}_{\text{out}}\sum\sum\underbrace{(I-Z_i)}_{\text{IN}}\underbrace{(I-Z_j)}_{\text{IN}}$$

$x_1 + x_2$
$x_1 + x_3$
$x_2 + x_3$
$N=3$

$x_1 + x_2$
$x_1 + x_3$
$x_1 + x_4$
$x_2 + x_3$
$x_2 + x_4$
$x_3 + x_4$
$N=4$

$$= \tfrac{1}{4}\sum\sum\underbrace{(I^2 - IZ_j - Z_i I + Z_i Z_j)}_{\text{FOIL}}$$

$$= \tfrac{1}{4}\sum\sum I - \tfrac{1}{4}\underbrace{\sum\sum(Z_i + Z_j)}_{\text{collect expand.}} + \tfrac{1}{4}\sum\sum Z_i Z_j$$

$$= \tfrac{1}{4}\underbrace{\frac{N(N-1)}{2}}_{\text{prev. triangular}} I - \underbrace{\frac{N+1}{4}\sum Z_i}_{Z_i \text{ appears } 2(N-1) \text{ times}} + \tfrac{1}{4}\sum\sum Z_i Z_j$$

$$2A = \underbrace{\tfrac{1}{2}\frac{N(N-1)}{2} I}_{\tfrac{1}{2}TI} - \underbrace{\frac{N+1}{2}}_{-2\frac{N-1}{2} = -(N-1) = +(1-N)}\sum Z_i + \tfrac{1}{2}\sum\sum Z_i Z_j$$

$$B = \sum x_i = \sum \tfrac{1}{2}(I - Z_i) = \tfrac{1}{2}\sum(I - Z_i)$$

$$= \tfrac{1}{2}\sum I - \tfrac{1}{2}\sum Z_i$$

$$= \frac{N}{2} I - \tfrac{1}{2}\sum Z_i$$

$$(1-N)B = \underbrace{\frac{N(1-N)}{2} I}_{-TI} - \frac{1-N}{2}\sum Z_i$$

In terms of the Z basis, our Ising model is thus

$$\tfrac{1}{2}\frac{N(N-1)}{2} I - \frac{N+1}{2}\sum Z_i + \tfrac{1}{2}\sum\sum Z_i Z_j - \frac{N(N-1)}{2} I - \frac{1-N}{2}\sum Z_i + \frac{N^2}{4}$$

$$\underbrace{\left(\tfrac{1}{2} - 1\right)}\frac{N(N-1)}{2} I - \underbrace{\left(\frac{N+1}{2} + \frac{1-N}{2}\right)}_{N/2}\sum Z_i + \tfrac{1}{2}\sum\sum Z_i Z_j + \frac{N^2}{4}$$

$$-\tfrac{1}{2}\frac{N(N-1)}{2} \to -\frac{N^2 - N}{4} = \frac{-N^2}{4} + \frac{N}{4}$$

There will be $\frac{N(N-1)}{2}$ edges. At least $N-2$ edges will be used for the graph. They are used to test connectivity. Only connections within groups can be used – no between-group connections nor disconnections can be used.

The ansatz must be changed to accommodate more qubits. There will be $N + \frac{N(N-1)}{2} = N + \frac{N^2}{2} - \frac{N}{2} = \frac{N(N+1)}{2}$ qubits. Each edge is indexed by two numbers with the second strictly greater than the first. The entries of the H matrix is 1 iff $(u,v) \notin E$. Also, for every pair, it contributes to the penalty if $x_u \neq x_v$. This means no disconnection nor intergroup edges can be used.

$$\sum\sum [(u,v) \notin E] e_{ij}$$

$$\sum\sum [x_u \neq x_v] e_{ij} = \sum [1 - x_u - x_v + 2x_u x_v] e_{ij}.$$

$$= \sum e_{ij} - \sum x_i e_{ij} - \sum x_j e_{ij} + \underbrace{\sum x_i x_j e_{ij}}_{\text{cubic.}}$$

$$\forall u \sum_v e_{uv} \geq 1$$